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Ideals Generated by Pfaffians

TADEUSZ JÓZEFIAK

Institute of Mathematics, Polish Academy of Sciences, Chopina 12, 87-100 Toruń, Poland

AND

PIOTR PRAGACZ

Institute of Mathematics, Nicholas Copernicus University, Chopina 12, 87-100 Toruń, Poland

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Influenced by the structure theorem of Buchsbaum and Eisenbud for Gorenstein ideals of depth 3, [4], we study in this paper ideals generated by Pfaffians of given order of an alternating matrix.

Let X be an n by n alternating matrix (i.e., $x_{ij} = -x_{ji}$ for $i < j$ and $x_{ii} = 0$) with entries in a commutative ring R with identity. One can associate with X an element $Pf(X)$ of R called Pfaffian of X (see [1] or [2] for the definition). For n odd $\det X = 0$ and $Pf(X) = 0$; for n even $\det X$ is a square in R and $Pf(X)^2 = \det X$. For a basis-free account and basic properties of Pfaffians we send interested readers to Chapter 2 of [4] or to [10].

For a sequence $i_1, \ldots, i_k$, $1 \leqslant i_r \leqslant n$, the matrix obtained from X by omitting rows and columns with indices $i_1, \ldots, i_k$ is again alternating; we write $Pf^{i_1,\ldots,i_k}(X)$ for its Pfaffian and call it the $(n-k)$-order Pfaffian of X. We are interested in ideals $Pf_{2p}(X)$ generated by all the $2p$-order Pfaffians of X, $0 \leqslant 2p \leqslant n$.

After some preliminaries in Section 1 we prove in Section 2 that the height and the depth of $Pf_{2p}(X)$ are bounded by the number

$$\mu(p, n) = (n - 2p + 1)(n - 2p + 2)/2.$$

Moreover this number is attained by the depth (and hence by the height) if the entries of X are algebraically independent over a subring of R, and under some flatness hypothesis.

In Section 3 we investigate more closely the ideal $Pf_{n-2}(X)$ for n even. We construct a free complex $\mathbf{L}(X)$ of length 6 with $H_0(\mathbf{L}(X)) = R/Pf_{n-2}(X)$ which gives a free resolution of $Pf_{n-2}(X)$ when depth $Pf_{n-2}(X) = 6$ and 2 is invertible

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in R. Since $\mu(n-1, n) = 6$ we obtain that $Pf_{n-2}(X)$ is perfect and even Gorenstein (when of depth 6).

As an application we get a free resolution of an ideal defining the Grassmannian $G(2, 6)$ of 2-dimensional linear subspaces in a 6-dimensional space.

1. Preliminaries

We will need in the sequel the following Laplace expansion type formula for Pfaffians.

LEMMA 1.1. *If j is a fixed integer, $1 \leqslant j \leqslant n$, then*

$$Pf(X) = \sum_{i<j} (-1)^{i+j-1} x_{ij} Pf^{ij}(X) + \sum_{i>j} (-1)^{i+j} x_{ij} Pf^{ij}(X).$$

LEMMA 1.2. *If x_{12} is invertible in R, then there exists an invertible matrix C such that*

(1) $${}^tCXC = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & \\ \hline \multicolumn{2}{c|}{0} & X' \end{array}\right);$$

(2) *the $n-2$ by $n-2$ matrix X' is alternating and*

$$x'_{ij} = x_{ij} + \frac{x_{1j}x_{2i} - x_{1i}x_{2j}}{x_{12}}, \qquad i, j = 3, \ldots, n,$$

(3) $Pf_{2p}(X) = Pf_{2(p-1)}(X')$ *for $p \geqslant 1$.*

Proof. We define

$$C = \left(\begin{array}{cc|ccc} x_{12}^{-1} & 0 & x_{23}x_{12}^{-1} & \cdots & x_{2n}x_{12}^{-1} \\ 0 & 1 & -x_{13}x_{12}^{-1} & \cdots & -x_{1n}x_{12}^{-1} \\ \hline & & 1 & & 0 \\ \multicolumn{2}{c|}{0} & & \ddots & \\ & & 0 & & 1 \end{array}\right).$$

One proves (1) and (2) by simple calculations and (3) using Lemma 1.1.

COROLLARY 1.3. *Let R be a local ring. If $Pf_{n-2}(X) = R$, then there exists an invertible matrix C over R such that*

$$(\#) \qquad {}^tCXC = \begin{bmatrix} 0 & 1 & & & & & & 0 \\ -1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & 1 & & & \\ & & & -1 & 0 & & & \\ & & & & & & 0 & a \\ & 0 & & & & & -a & 0 \end{bmatrix}$$

for some $a \in R$.

Proof. If $Pf_{n-2}(X) = R$, then at least one entry of X is invertible. By Lemma 1.2 (possibly after rearrangement of some rows and the same columns of X) we get an $n-2$ by $n-2$ alternating matrix X' satisfying (1)–(3) of the lemma. If $n-2=2$ we are done. If not, we proceed by induction with respect to n.

2. HEIGHT AND DEPTH OF $Pf_{2p}(X)$

THEOREM 2.1. *Let R be a commutative Noetherian ring with identity and let X be an n by n alternating matrix with entries in R. Then every minimal prime ideal of the ideal $Pf_{2p}(X)$ has height at most equal to $\mu(p, n)$.*

Proof. We only sketch the proof because its idea is similar to that in [5, pp. 202–203].

We argue by induction on n. If $n \leqslant 2$ or $p = 1$ the theorem follows from the generalized principal ideal theorem of Krull.

Suppose $n > 2$, $p > 1$ and let P be a minimal prime ideal of $Pf_{2p}(X)$. One may assume that R is local with maximal ideal P and that $Pf_{2p}(X)$ is P-primary. If at least one entry of X is invertible in R, then by Lemma 1.2 $Pf_{2p}(X) = Pf_{2p-2}(X')$ for some $n-2$ by $n-2$ alternating matrix X' over R. By the induction hypothesis ht $P \leqslant \mu(p-1, n-2) = \mu(p, n)$.

If all the entries of X belong to P, then we consider an alternating n by n matrix

$$\bar{X} = X + \left(\begin{array}{cc|c} 0 & t & \\ -t & 0 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right)$$

over a polynomial ring $R[t]$, where t is an indeterminate over R. For arbitrary alternating submatrix B of X and the corresponding $\bar{B}$ of $\bar{X}$ there exists $h \in R[t]$ such that $Pf(\bar{B}) = Pf(B) + th$. This proves the equality

$$(Pf_{2p}(X), t)\, R[t] = (Pf_{2p}(\bar{X}), t)\, R[t]$$

and allows us to apply the method on p. 203 of [5].

COROLLARY 2.2. *Under the hypothesis of Theorem 2.1 we have an inequality depth $Pf_{2p}(X) \leqslant \mu(p, n)$.*

The equality holds in some cases, as the next theorem shows.

THEOREM 2.3. *Let S be a commutative Noetherian ring with identity, R a Noetherian subring of S with the same identity. Let $\{x_{ij}\}$, $1 \leqslant i < j \leqslant n$, be a sequence of elements of S which are algebraically independent over R. Assume that S is flat as an algebra over $R[\{x_{ij}\}]$. If we put $x_{ji} = -x_{ij}$ for $i < j$ and $x_{ii} = 0$, and define $X = (x_{ij})$, then depth $Pf_{2p}(X) = \mu(p, n)$.*

For the proof of the theorem we need the following simple lemma.

LEMMA 2.4. *Let R be a subring of a commutative ring S with identity and let $x_1, \ldots, x_q$ be a sequence of elements in S which are algebraically independent over R. Assume that t is a nonzero divisor in S belonging to $R[x_1, \ldots, x_s]$, $s < q$, and write $\bar{R} = R[x_1, \ldots, x_s]_{\{t^k\}}$, $\bar{S} = S_{\{t^k\}}$ for the localizations of the corresponding rings at the powers of t; moreover let $\bar{a}_{s+1}, \ldots, \bar{a}_q$ be elements of $\bar{R}$. Then $\bar{R} \subset \bar{S}$ and the elements $\bar{x}_{s+1} - \bar{a}_{s+1}, \ldots, \bar{x}_q - \bar{a}_q$ are algebraically independent over $\bar{R}$.*

Proof of Theorem 2.3. If $p = 1$, then $Pf_2(X) = (x_{ij})S$. Since S is flat over $R[\{x_{ij}\}]$ the sequence $\{x_{ij}\}$, $i < j$, is S-regular and therefore depth $Pf_2(X) = \mu(1, n)$.

Now we argue by induction on n, assuming $n > 1$, $p > 1$. Let $v_1, \ldots, v_m$ be a maximal S-regular sequence contained in $Pf_{2p}(X)$. By Corollary 2.2 we know that $m \leqslant \mu(p, n)$, hence in view of $p > 1$ we have $m < (n^2 - n)/2$. Let $J = (v_1, \ldots, v_m)$ for short. Since $Pf_{2p}(X)$ consists of zero divisors on J, there exists a prime P associated with J and containing $Pf_{2p}(X)$. Thus $m =$ depth $J =$ depth $Pf_{2p}(X) =$ depth P. The inequality $m < (n^2 - n)/2$ implies $x_{ij} \notin P$ for some i, j. Without loss of generality we can assume that $x_{12} \notin P$. We write $\bar{R} = R[x_{12}, \ldots, x_{1n}, x_{23}, \ldots, x_{2n}]_{\{x_{12}^k\}}$, $\bar{S} = S_{\{x_{12}^k\}}$, and denote by $\bar{v}$ (respectively $\bar{I}$) the image of an element v (respectively, an ideal I) under the natural homomorphism $S \to \bar{S}$. We have $\bar{J} \subset \overline{Pf_{2p}(X)} \subset \bar{P}$, $\bar{P}$ is proper and depth $\bar{J} = m$ since $\bar{J}$ is generated by an $\bar{S}$-regular sequence $\bar{v}_1, \ldots, \bar{v}_m$. On the other hand, $\bar{P}$ is associated prime of $\bar{J}$ because $x_{12} \notin P$. Therefore $m =$ depth $\bar{J} =$ depth $\overline{Pf_{2p}(X)} =$ depth $\bar{P}$. Observe that $\overline{Pf_{2p}(X)} = Pf_{2p}(\bar{X})$ where $\bar{X} = (\bar{x}_{ij})$.

Since $\bar{x}_{12}$ is invertible in $\bar{S}$ we may apply Lemma 1.2 to $\bar{S}$ and $\bar{X}$, and obtain $\overline{Pf}_{2p}(X) = Pf_{2(p-1)}(\tilde{X})$ where $\tilde{X} = (\bar{x}_{ij} - \bar{a}_{ij})$ is the alternating $n-2$ by $n-2$ matrix with entries in $\bar{S}$, $\bar{a}_{ij} \notin R$, $3 \leqslant i, j \leqslant n$. By Lemma 2.4 we infer that the elements $\{\bar{x}_{ij} - \bar{a}_{ij}\}$, $3 \leqslant i, j \leqslant n$, are algebraically independent over $\bar{R}$. Moreover $\bar{R}[\{\bar{x}_{ij} - \bar{a}_{ij}\}]$, $3 \leqslant i, j \leqslant n$, is equal to $R[\{x_{ij}\}]_{\{x_{12}^k\}}$, $1 \leqslant i < j \leqslant n$, and $\bar{S}$ is flat over $R[\{x_{ij}\}]_{\{x_{12}^k\}}$. Hence by the induction hypothesis we finally get $m = \operatorname{depth} \overline{Pf}_{2p}(X) = \operatorname{depth} Pf_{2(p-1)}(\tilde{X}) = \mu(p-1, n-2) = \mu(p, n)$.

COROLLARY 2.5. *Let R be a commutative Noetherian ring and $S = R[\{x_{ij}\}]$, $1 \leqslant i < j \leqslant n$, a polynomial ring over R in $(n^2 - n)/2$ indeterminates $\{x_{ij}\}$. Then depth $Pf_{2p}(X) = \mu(p, n)$ where X is an alternating matrix determined by $\{x_{ij}\}$.*

Remark 2.6. Corollary 2.5 was proved independently by Kleppe in [8] if R is a field.

COROLLARY 2.7. *Let S be a local algebra over a field K and let $\{x_{ij}\}$, $1 \leqslant i < j \leqslant n$, be a regular sequence in S. Then depth $Pf_{2p}(X) = \mu(p, n)$ for the alternating matrix $X = (x_{ij})$.*

Proof. By [6] $\{x_{ij}\}$ are algebraically independent over K and S is flat over $K[\{x_{ij}\}]$. Therefore the corollary follows immediately from the theorem.

3. A FREE RESOLUTION OF $Pf_{n-2}(X)$

Let R be a commutative ring with identity and $X = (x_{ij})$ an alternating matrix of even degree n over R. We write $Y = (y_{ij})$ for the following alternating matrix:

$$\begin{aligned} y_{ij} &= (-1)^{i+j} Pf^{ij}(X) && \text{if } i < j \\ &= 0 && \text{if } i = j \\ &= (-1)^{i+j-1} Pf^{ij}(X) && \text{if } i > j. \end{aligned}$$

By Lemma 1.1 we have $XY = YX = Pf(X)E$ where E is n by n identity matrix. We are fixing the matrix X (and hence Y) in what follows.

Let $M_n(R)$ be the free R-module of all n by n matrices over R and $A_n(R)$ (respectively, $S_n(R)$) the free submodule of $M_n(R)$ consisting of all alternating (respectively, symmetric) matrices. Furthermore, let tr: $M_n(R) \to R$ denote the trace map.

We define a free complex $\mathbf{L}(X)$ of length 6 associated with $Pf_{n-2}(X)$:

$$L_0 = R, \quad L_1 = M_n(R)/S_n(R), \quad L_2 = \operatorname{Ker}(M_n(R) \xrightarrow{\text{tr}} R),$$

$$L_3 = (M_n(R)/A_n(R)) \oplus S_n(R), \quad L_4 = \operatorname{Coker}(R \xrightarrow{\text{Id}} M_n(R)),$$

$$L_5 = A_n(R), \quad L_6 = R,$$

and differentials are as follows

$$
\begin{aligned}
d_1(M \bmod S_n(R)) &= \operatorname{tr}(YM),\\
d_2(N) &= XN \bmod S_n(R),\\
d_3(P \bmod A_n(R), Q) &= Y(P + {}^tP) - QX,\\
d_4(C \bmod RE) &= (XC \bmod A_n(R),\ CY + {}^t(CY)),\\
d_5(B) &= BX \bmod RE,\\
d_6(r) &= rY.
\end{aligned}
$$

It is easy to verify that $\mathbf{L}(X)$ is well-defined complex of free R-modules. Observe that $H_0(\mathbf{L}(X)) = R/Pf_{n-2}(X)$.

LEMMA 3.1. *The complex* $\mathbf{L}(X)$ *is self-dual, i.e., there exists an* R*-isomorphism* $\varphi = \{\varphi_i\}: \mathbf{L}(X) \to \mathbf{L}(X)^*$ *of complexes, where* $\mathbf{L}(X)^*$ *is the dual complex of* $\mathbf{L}(X)$. *Maps* φ_i, $i = 0, 1, \dots, 6$, *can be defined in the following way*:

(1) $\varphi_0(r)$ *and* $\varphi_6(r)$ *are multiplication by* $-r$ *and* r, *respectively*,

(2) φ_1, φ_5 *are induced by the pairing* $A_n(R) \otimes (M_n(R)/S_n(R)) \to R$, $(A, B \bmod S_n(R)) \mapsto \operatorname{tr}(AB)$, $\varphi_5(B)(\cdot) = \operatorname{tr}(B\,\cdot)$, $\varphi_1(M \bmod S_n(R))(\cdot) = -\operatorname{tr}(M\,\cdot)$,

(3) φ_2, φ_4 *are induced by the pairing* $\operatorname{Ker}(M_n(R) \xrightarrow{\mathrm{tr}} R) \otimes \operatorname{Coker}(R \xrightarrow{\mathrm{Id}} M_n(R)) \to R$, $(M, N \bmod RE) \mapsto \operatorname{tr}(MN)$, $\varphi_4(C \bmod RE)(\cdot) = \operatorname{tr}(C\,\cdot)$, $\varphi_2(N)(\cdot) = -\operatorname{tr}(N\,\cdot)$,

(4) $\varphi_3(P \bmod A_n(R), Q) = \operatorname{tr}(Q\,\cdot) \oplus (-\operatorname{tr}(P\,\cdot))$.

Now we can state the main result of this section.

THEOREM 3.2. *Let* R *be a commutative Noetherian ring with identity and let* 2 *be invertible in* R. *Moreover, let* $X = (x_{ij})$ *be an* n *by* n *alternating matrix with entries in* R. *If depth* $Pf_{n-2}(X) = 6$ (*the largest possible*), *then the complex* $\mathbf{L}(X)$ *is acyclic and is a free resolution of* $R/Pf_{n-2}(X)$.

The proof of the theorem requires several preliminary lemmas.

LEMMA 3.3. *Let* $\varphi: R \to R'$ *be a ring homomorphism,* $X = (x_{ij})$ *an alternating matrix over* R, *and* $X' = (\varphi(x_{ij}))$. *Then the complexes* $\mathbf{L}(X) \otimes_R R'$ *and* $\mathbf{L}(X')$ *are isomorphic over* R'.

LEMMA 3.4. *The complexes* $\mathbf{L}({}^tCXC)$ *and* $\mathbf{L}(X)$ *are isomorphic for an arbitrary invertible* n *by* n *matrix* C.

Proof. Let F be a free R-module of rank n and let F^* be the dual module of F. A map $f: F^* \to F$ is said to be alternating if with respect to some (and therefore every) basis and dual basis of F and F^*, the matrix of f is alternating. There is an element $\varphi \in F \otimes F$ which corresponds to f by the canonical isomorphisms $\mathrm{Hom}(F^*, F) \simeq (F^*)^* \otimes F \simeq F \otimes F$. f is alternating if and only if φ belongs to the image of the monomorphism $\wedge^2 F \to F \otimes F$, $x \wedge y \mapsto x \otimes y - y \otimes x$.

We are going to prove the lemma by assigning to a pair (f, e^*), where $f: F^* \to F$ is an alternating map and e^* a free generator of $\wedge^n F^*$, a free complex $\mathbf{K}(f, e^*)$ of length 6 and showing that $\mathbf{L}(X)$ and $\mathbf{L}({}^tCXC)$ are both isomorphic with $\mathbf{K}(f, e^*)$.

The components of $\mathbf{K} = \mathbf{K}(f, e^*)$ are as follows:

$$K_0 = R, \qquad K_1 = \wedge^2 F, \qquad K_2 = \mathrm{Ker}(F^* \otimes F \xrightarrow{\mathrm{ev}} R),$$
$$K_3 = S_2(F) \oplus D_2(F^*), \qquad K_4 = \mathrm{Coker}(R \xrightarrow{\mathrm{Id}} \mathrm{Hom}(F, F) \simeq F^* \otimes F),$$
$$K_5 = \wedge^2 F^*, \qquad K_6 = R,$$

where ev stands for the evaluation map and $\wedge^2$, S_2, D_2 are the second exterior, symmetric, and divided power functors, respectively.

To define differentials we recall that the exterior algebra $\wedge F$ possesses a unique system of divided powers (see [4, 10] for details). Moreover divided powers $\varphi^{(p)}$ of φ are related to Pfaffians by the formula

$$\varphi^{(p)} = \sum_{|H|=2p} Pf(X_H)\, e_H ,$$

where $\{e_H\}$ is the standard basis of $\wedge F$ associated with a basis $\{e_i\}$ of F, X is the matrix of f with respect to $\{e_i\}$ and dual basis $\{e_i^*\}$, and X_H denotes the principal submatrix of X determined by the set H.

We need the fact that $\wedge F$ and $\wedge F^*$ are modules over each other and we adopt Buchsbaum and Eisenbud notation from [4] writing $a(b)$ for the result of an operation of $a \in \wedge F$ on $b \in \wedge F^*$ and vice versa; therefore $a(b) \in \wedge F^*$ and $b(a) \in \wedge F$.

With the pair (f, e^*) we associate a map $g = g(f, e^*): F \to F^*$ by putting

$$g(a) = (-1)^{(n-2)/2} a(\varphi^{((n-2)/2)}(e^*)).$$

If $e^* = e_1^* \wedge \cdots \wedge e_n^*$, then Y is a matrix of g with respect to $\{e_i\}$ and $\{e_i^*\}$.

Now we are ready to define differentials $\partial = \{\partial_i\}$ on $\mathbf{K}$:

$$\partial_1 \text{ is equal to } \wedge^2 F \to F \otimes F \xrightarrow{g \otimes 1} F^* \otimes F \xrightarrow{\mathrm{ev}} R,$$
$$\partial_2(u \otimes w) = f(u) \wedge w,$$
$$\partial_3 \mid S_2(F): \qquad u \vee w \mapsto g(u) \otimes w + g(w) \otimes u,$$
$$\partial_3 \mid D_2(F^*): u^{(2)} \mapsto -u \otimes f(u),$$

∂_4 is induced by the composition

$$F^* \otimes F \xrightarrow{(f\otimes 1, -1\otimes g)} F \otimes F \oplus F^* \otimes F^* \to S_2(F) \oplus D_2(F^*),$$

∂_5 is induced by $\wedge^2 F^* \to F^* \otimes F^* \xrightarrow{-1\otimes f} F^* \otimes F$,

$$\partial_6(r) = (-1)^{n/2} \varphi^{((n-2)/2}(e^*)r.$$

It is easy to check that $\mathbf{L}(X) \simeq \mathbf{K}(f, e_1^* \wedge \cdots \wedge e_n^*)$ where X is the matrix of f with respect to $\{e_i\}$ and $\{e_i^*\}$. Let C be an n by n invertible matrix and let $\{f_i\}$ be a basis of F corresponding to a change of $\{e_i\}$ by means of C; furthermore, let $\{f_i^*\}$ be the dual basis of $\{f_i\}$. Then the matrix of f with respect to $\{f_i\}$ and $\{f_i^*\}$ is equal to ${}^tCXC$ and therefore $\mathbf{L}({}^tCXC) \simeq \mathbf{K}(f, f_1^* \wedge \cdots \wedge f_n^*)$. But $f_1^* \wedge \cdots \wedge f_n^* = r(e_1^* \wedge \cdots \wedge e_n^*)$ for an invertible element r of R. Thus it is enough to show that $\mathbf{K}(f, e^*) \simeq \mathbf{K}(f, re^*)$ for r invertible. We define $\psi = \{\psi_i\}: \mathbf{K}(f, e^*) \to \mathbf{K}(f, re^*)$ as follows: ψ_0 is the multiplication by r^3, ψ_1, ψ_2 by r^2, $\psi_3(u, w) = (ru, r^2 w)$, ψ_4, ψ_5 are the multiplications by r, $\psi_6 = \mathrm{Id}$. It is straightforward to check that ψ is an isomorphism of the complexes in question.

Proof of Theorem 3.2. By the acyclicity lemma of Peskine and Szpiro [9] it is enough to prove that $\mathbf{L}(X)_P$ is exact for every prime P with depth $P < 6$. Since depth $Pf_{n-2}(X) = 6$ we infer that $Pf_{n-2}(X) \not\subset P$ for such a P, and hence $Pf_{n-2}(X_P) = R_P$ where X_P is the matrix X considered over R_P. By Lemma 3.3 $\mathbf{L}(X)_P \simeq \mathbf{L}(X_P)$ so it suffices to prove the theorem for R local and X with $Pf_{n-2}(X) = R$. Applying Corollary 1.3 and Lemma 3.4 to such a matrix X we see that it is enough to prove the exactness of the complex $\mathbf{L}$ for a matrix of the form (#) (see Corollary 1.3). The corresponding Y equals

$$\begin{bmatrix} 0 & -a & & & & & \\ a & 0 & & & & 0 & \\ & & \ddots & & & & \\ & & & 0 & -a & & \\ & & & a & 0 & & \\ & 0 & & & & 0 & -1 \\ & & & & & 1 & 0 \end{bmatrix}.$$

By the self-duality of $\mathbf{L}(X)$ (Lemma 3.1) we only need to prove that

$$\operatorname{Ker} d_1 = \operatorname{Im} d_2, \qquad \operatorname{Ker} d_2 = \operatorname{Im} d_3, \qquad \operatorname{Ker} d_3 = \operatorname{Im} d_4. \qquad (\#\#)$$

It is in the proof of the last two equalities in (##) that we use the hypothesis that 2 is invertible in R. The complex $\mathbf{L}(X)$ is not exact if R is of characteristic 2.

$\operatorname{Ker} d_1 = \operatorname{Im} d_2$. It is easy to check that both $\operatorname{Im} d_2$ and $\operatorname{Ker} d_1$ are equal to the free submodule of $M_n(R)/S_n(R)$ with basis $\{E_{2m-1,2m} - aE_{n-1,n}\}$, $m =$

$1, \ldots, (n-2)/2$, $\{E_{ij}\}$, $i < j$, $(i, j) \neq (2m-1, 2m)$, $m = 1, \ldots, n/2$, where $\{E_{ij}\}$ denotes the standard basis of $M_n(R)$.

An arbitrary matrix A of $M_n(R)$ can be written uniquely in the form

$$\left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right)$$

where A_1, A_2, A_3, A_4 are $n-2$ by $n-2$, $n-2$ by 2, 2 by $n-2$ and 2 by 2 matrices, respectively.

Ker d_2 = Im d_3. If XN is symmetric with N of trace zero we define $Q_1 = N_1X_1$, $Q_2 = Q_3 = Q_4 = 0$, $P_1 = P_2 = 0$, $P_3 = -Y_4N_3$, $P_4 = -1/2Y_4N_4$. One can check that Q is symmetric and $d_3(P \bmod A_n(R), Q) = N$.

Ker d_3 = Im d_4. Suppose that $Y(P + {}^tP) = QX$ for Q symmetric. We define a matrix C by putting $C_1 = -X_1P_1$, $C_2 = -Q_2Y_4$, $C_3 = 0$, $C_4 = -1/2Q_4Y_4$; then $d_4(C \bmod RE) = (P \bmod A_n(R), Q)$.

The equality Ker d_3 = Im d_4 can be proved also using the criterion formulated in Lemma 2 of [3], self-duality of $\mathbf{L}(X)$, and the remaining equalities in (##).

To state the next corollary we recall that an ideal I in a local ring R is Gorenstein if

(a) $\operatorname{depth} I = pd_R(R/I) = p$,

(b) $\operatorname{Ext}_R{}^p(R/I, R) \simeq R/I$.

In another language this is equivalent to the existence of a free resolution of R/I over R of length p = depth I with the last (pth) term isomorphic to R, [4]. If R is regular, then I is a Gorenstein ideal if and only if R/I is a Gorenstein ring.

From Theorem 3.2 we get

COROLLARY 3.6. *If R is a local ring and 2 is invertible in R, then $Pf_{n-2}(X)$ is a Gorenstein ideal for every alternating matrix X over R such that depth $Pf_{n-2}(X) = 6$.*

Remark 3.7. It has been recently proved (see [7]) that under certain generic hypotheses all Pfaffians in characteristic zero are Gorenstein. Moreover we have been informed by D. Laksov that the characteristic zero assumption can be dropped.

Let K be a field and let $G(2, 6)$ be the Grassmannian of all 2-dimensional linear subspaces in K^6. The ideal defining $G(2, 6)$ is equal to $Pf_4(T)$ where $T = (t_{ij})$ is 6 by 6 alternating matrix over a polynomial ring $R = K[\{t_{ij}\}]$, $1 \leqslant i < j \leqslant 6$. Therefore from Theorem 3.2 and Corollary 2.5 follows

COROLLARY 3.8. *If* char $K \neq 2$, *then $\mathbf{L}(T)$ is an R-free resolution of R/I where I denotes the ideal of $G(2, 6)$.*

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